

CONTRACT TECHNICAL REPORT

*"A CONTINUUM MODEL FOR VISCOELASTICITY,
DAMAGE, AND PERMANENT DEFORMATION WITH
APPLICATION TO SWIRL-MAT POLYMERIC
COMPOSITES"*

By

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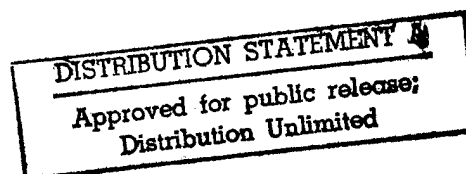
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A CONTINUUM MODEL FOR VISCOELASTICITY, DAMAGE, AND PERMANENT DEFORMATION WITH APPLICATION TO SWIRL-MAT POLYMERIC COMPOSITES

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ABSTRACT

A comprehensive thermodynamic framework is proposed for modeling the behavior of polymeric composites exhibiting viscoelasticity, damage and permanent deformation. In the present formulation, the total strain is decomposed into viscoelastic and permanent components. The effect of damage on the constitutive behavior is introduced through direct coupling between damage and the internal state variables representing deformation of the undamaged (virgin) material. This approach accounts for time-dependent damage as well as damage-induced changes in material symmetry. The viscoelastic part of the proposed constitutive model is first formulated for a general case of damage coupled with nonlinear viscoelasticity, and subsequently specialized to coupling with linear viscoelasticity. Permanent deformations are discussed in view of recent experimental findings. Finally, the proposed constitutive model is applied to experimental data for swirl-mat polymeric composites.

1. INTRODUCTION

There is a growing interest in the use of polymeric composites for structural applications. Some reasons for such growing interest are the significantly lighter weight of these materials and the ease with which they can be manufactured. In all applications durability plays a crucial role. For instance, since the expected life of an automotive vehicle is roughly fifteen years, polymeric composites intended for use in automotive applications are required to endure for the same length of time. It is well-known that polymeric composites containing relatively small fractions of brittle fibers (roughly 30% by volume) creep viscoelastically (e.g., Jerina *et al.*, 1979). In addition, experimental investigations (e.g., Suvorova, 1985; Schapery, 1989; Maire, 1992; Smith and Weitsman, 1996) indicate that polymeric composites undergo both damage and permanent deformation. The above features of material behavior are all life-limiting factors in that they can cause excessive dimensional distortions and stiffness and residual strength degradations. The aim of this paper is to establish a framework for modeling the foregoing features of material behavior. Such a framework is the basis for performing durability analyses, which play a major role in engineering design.

Damage observed in polymeric composites may occur in the form of microcracks along fiber bundles and also microdebondings along the fiber-matrix interface (e.g. Smith and Weitsman, 1996). Up to the present time, modeling such microcrack damage processes has been concerned primarily with brittle materials (e.g. rocks and concrete), exhibiting essentially elastic behavior, and metals exhibiting plastic or creep behavior (e.g. Lemaitre, 1984; Chaboche, 1988; Krajcinovic, 1989; Ju, 1989; Hansen and Schreyer, 1994). A common feature in these works is that the effect of damage on material behavior is introduced through the so-called effective stress. The notion of effective stress was first proposed for one-dimensional creep problems by Kachanov (1958), who introduced the effective stress as the applied stress magnified by a scalar factor representing the ratio between the original (intact) load carrying area and the reduced area due to damage. However, it is well-known that microcrack damage also affects the material symmetry. To account for this, the effective stress was subsequently introduced through a tensorial mapping of the applied stress.

Despite the progress made in modeling microcrack damage processes, less attention has been paid to modeling such processes in viscoelastic materials (e.g. plastics and polymeric composites). Notably, Schapery (1981, 1994, 1996) established the basic formulation for viscoelastic response that is accompanied by microstructural changes, such as profuse micro-

cracking. These formulations account for viscoelasticity by means of a set of scalar valued internal state variables similar to that employed in this article. The microstructural changes are represented in Schapery's work by means of another set of internal state variables whose evolutionary laws are motivated by considerations of viscoelastic fracture mechanics. An additional set of internal state variables is also employed by Schapery to account for permanent (non-recoverable) deformation. It may also be mentioned that Weitsman (1988) attempted to incorporate the effect of damage by coupling viscoelasticity and damage for a special class of linear viscoelastic materials.

In this work, the formulation of coupled viscoelastic and damage response is patterned on the methodology of continuum damage mechanics (e.g. Lemaitre and Chaboche, 1985; Krajcinovic, 1989; Lemaitre, 1992). The formulation employs several of Schapery's concepts as well as the thermodynamics framework outlined by Lubliner (1972), but is cast within the context of continuum damage mechanics. This format is correlated with damage induced changes in material symmetry and results in expressions incorporating effective stresses.

An additional issue considered in this work is the permanent deformation that accompanies the viscoelastic damage behavior of polymeric composites. In general, this permanent deformation increases with the load as well as creep duration (e.g. Tuttle *et al.*, 1995 ; Smith and Weitsman, 1996). The physical origin for permanent deformation in polymeric composites arises from several possible sources. For instance, the ability of the polymeric matrix itself to undergo irreversible structural changes may lead to non-recoverable strains. Permanent strains could also be due to frictional sliding following fiber-matrix interfacial microdebondings, as well as internal surface roughness and other irregularities resisting microcrack closure upon unloading.

In Section 2 of this article, we present a general thermodynamics framework that accounts for viscoelasticity, damage and permanent deformation. We proceed by modeling the coupling between viscoelasticity and damage in Section 3. In Section 4 permanent deformation is considered briefly but in a general manner. In Section 5, the model is applied to a type of so-called swirl-mat polymeric composites. Section 6 concludes with some general remarks pertinent to the present work and also contains some specific observations relevant to the material considered in Section 5.

2. GENERAL THERMODYNAMICS FRAMEWORK

Consider a polymeric composite material and let γ_r ($r = 1, 2, \dots, R$) denote R scalar valued internal state variables representing the internal degrees of freedom of molecular motion in the polymeric matrix. Also, let ζ_n ($n = 1, 2, \dots, N$) denote N scalar valued internal state variables associated with permanent deformation¹. The internal state variable representing damage is taken in the form a symmetric fourth rank tensor ω_{ijkl} which suffices to account for general damage-induced changes in material symmetry. The formulation can be readily modified to accommodate damage variables of other tensorial ranks. Throughout this paper the subscripts r and q are reserved for viscoelastic quantities; m and n for quantities associated with permanent deformation; and $a, b, c, d, i, j, k, l, p$ and s are associated with tensorial quantities and cover the range 1, 2, 3. Also, the summation convention is implied over the range of repeated indices unless stated otherwise.

Viscoelasticity, damage and permanent deformation are all forms of irreversible thermodynamic processes. For a closed system and small strain formulation, the entropy production inequality can be written in the form (Coleman and Gurtin, 1967)

$$-\dot{\phi} - \varepsilon_{ij} \dot{\sigma}_{ij} - S \dot{T} - \frac{q_i T_{,i}}{T} \geq 0, \quad (1)$$

where ϕ is the Gibb's free energy (per unit volume), ε_{ij} - components of a suitably defined volume average infinitesimal strain tensor, σ_{ij} - components of the Cauchy stress tensor, S - entropy (per unit volume), T - temperature, q_i - components of the heat flux vector, $T_{,i} = \partial T / \partial x_i$ - components of the temperature gradient, and x_i - space coordinates. Also, in (1) the overdot signifies differentiation with respect to time.

Consider a Gibb's free energy of the form

$$\phi = \phi(\sigma_{ij}, \gamma_r, \zeta_n, \omega_{abcd}, T).$$

The function ϕ is assumed to be continuous and sufficiently differentiable with respect to its arguments. Considerations of the entropy production inequality in (1) give the familiar relations

$$\varepsilon_{ij} = -\frac{\partial \phi}{\partial \sigma_{ij}}, \quad (2)$$

¹Here we adopt the variables selected by Lubliner (1990), though some of those variables may in fact be components of the second rank "back stress" tensor.

$$S = -\frac{\partial \phi}{\partial T}, \quad (3)$$

and

$$\Gamma_r \dot{\gamma}_r + \mathcal{Z}_n \dot{\zeta}_n + \Omega_{ijkl} \dot{\omega}_{ijkl} - \frac{q_i T_{,i}}{T} \geq 0, \quad (4)$$

where Γ_r , \mathcal{Z}_n and Ω_{ijkl} are the thermodynamic forces conjugate to the internal state variables γ_r , ζ_n and ω_{ijkl} , respectively, and are given by

$$\Gamma_r = -\frac{\partial \phi}{\partial \gamma_r}, \quad (5)$$

$$\mathcal{Z}_n = -\frac{\partial \phi}{\partial \zeta_n}, \quad (6)$$

and

$$\Omega_{ijkl} = -\frac{\partial \phi}{\partial \omega_{ijkl}}. \quad (7)$$

Guided by Lubliner (1972), we assume strain decomposition in the form

$$\varepsilon_{ij} = \varepsilon_{ij}^v(\sigma_{kl}, \gamma_r, \omega_{abcd}, T) + \varepsilon_{ij}^p(\zeta_n, \omega_{abcd}, T), \quad (8)$$

where ε_{ij}^v and ε_{ij}^p are components of the viscoelastic and permanent strain tensors, respectively. Since ϕ is a continuous function of σ_{ij} , then it follows from (2) that

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}. \quad (9)$$

Substitution of (8) into (9) yields

$$\frac{\partial \varepsilon_{ij}^v}{\partial \sigma_{kl}} = \frac{\partial \varepsilon_{kl}^v}{\partial \sigma_{ij}}. \quad (10)$$

Equation(10) implies that there exists a scalar valued function, say ϕ^v , such that

$$\varepsilon_{ij}^v = -\frac{\partial \phi^v}{\partial \sigma_{ij}}. \quad (11)$$

We denote by ϕ^v the Gibb's free energy associated with the viscoelastic deformation.

From (2), (8) and (11) it follows that ϕ can be written in the general form

$$\phi = \phi^v - \sigma_{ij} \varepsilon_{ij}^p + \mathcal{F}(\gamma_r, \zeta_n, \omega_{abcd}, T), \quad (12)$$

where \mathcal{F} is an arbitrary scalar valued function of its arguments. The physical significance of \mathcal{F} becomes clear when we consider the Helmholtz free energy ψ using the relation (Callen, 1960)

$$\phi(\sigma_{ij}, \gamma_r, \zeta_n, \omega_{abcd}, T) = \psi(\varepsilon_{ij}, \gamma_r, \zeta_n, \omega_{abcd}, T) - \sigma_{ij} \varepsilon_{ij}. \quad (13)$$

Following Lubliner (1972), we assume that ψ can be decomposed in the form

$$\psi = \psi^v(\varepsilon_{ij}^v, \gamma_r, \omega_{abcd}, T) + \psi^p(\zeta_n, \omega_{abcd}, T), \quad (14)$$

where ψ^v and ψ^p are the Helmholtz free energies associated with the viscoelastic and permanent deformations, respectively. Substituting (12) and (14) into (13)

$$\phi^v(\sigma_{ij}, \gamma_r, \omega_{abcd}, T) + \mathcal{F}(\gamma_r, \zeta_n, \omega_{abcd}, T) = \psi^v(\varepsilon_{ij}^v, \gamma_r, \omega_{abcd}, T) - \sigma_{ij} \varepsilon_{ij}^v + \psi^p(\zeta_n, \omega_{abcd}, T). \quad (15)$$

The Legendre transform of ψ^v (Callen, 1960) gives

$$\psi^v(\varepsilon_{ij}^v, \gamma_r, \omega_{abcd}, T) - \sigma_{ij} \varepsilon_{ij}^v = \mathcal{G}(\sigma_{ij}, \gamma_r, \omega_{abcd}, T), \quad (16)$$

where \mathcal{G} is a scalar valued function of its arguments. From (15) and (16), it is clear that $\mathcal{G} = \phi^v$ and $\mathcal{F} = \psi^p$; whereby \mathcal{F} is independent of γ_r . Thus (16) yields

$$\phi^v(\sigma_{ij}, \gamma_r, \omega_{abcd}, T) = \psi^v(\varepsilon_{ij}^v, \gamma_r, \omega_{abcd}, T) - \sigma_{ij} \varepsilon_{ij}^v, \quad (17)$$

and (12) reads

$$\phi = \phi^v - \sigma_{ij} \varepsilon_{ij}^p + \psi^p. \quad (18)$$

Since

$$-\frac{\partial}{\partial \sigma_{ij}}(-\sigma_{ij} \varepsilon_{ij}^p + \psi^p) = \varepsilon_{ij}^p,$$

equation (18) may be written as

$$\phi = \phi^v(\sigma_{ij}, \gamma_r, \omega_{abcd}, T) + \phi^p(\sigma_{ij}, \zeta_n, \omega_{abcd}, T), \quad (19)$$

where ϕ^p is given by

$$\phi^p = \psi^p - \sigma_{ij} \varepsilon_{ij}^p. \quad (20)$$

and

$$\varepsilon_{ij}^p = -\frac{\partial \phi^p}{\partial \sigma_{ij}}, \quad (21)$$

In view of (5) and (19), the thermodynamic forces Γ_r are given by

$$\Gamma_r = -\frac{\partial \phi^v}{\partial \gamma_r}. \quad (22)$$

Similarly, (6) and (19) yield

$$\mathcal{Z}_n = -\frac{\partial \phi^p}{\partial \zeta_n}, \quad (23)$$

Also, in view of (7) and (19) the thermodynamic forces Ω_{ijkl} can be decomposed in the form

$$\Omega_{ijkl} = \Omega_{ijkl}^v + \Omega_{ijkl}^p, \quad (24)$$

where

$$\Omega_{ijkl}^v = -\frac{\partial \phi^v}{\partial \omega_{ijkl}}, \quad (25)$$

and

$$\Omega_{ijkl}^p = -\frac{\partial \phi^p}{\partial \omega_{ijkl}}. \quad (26)$$

Finally, from the dissipation inequality (4), we have the following requirements

$$\Gamma_r \dot{\gamma}_r \geq 0, \quad (27)$$

$$\Gamma_r \dot{\gamma}_r + \Omega_{ijkl}^v \dot{\omega}_{ijkl} \geq 0, \quad (28)$$

$$\mathcal{Z}_n \dot{\zeta}_n \geq 0, \quad (29)$$

and

$$\mathcal{Z}_n \dot{\zeta}_n + \Omega_{ijkl}^p \dot{\omega}_{ijkl} \geq 0. \quad (30)$$

Inequality (27) should always be satisfied whenever viscoelastic deformation occurs. When viscoelastic deformation is accompanied by damage inequality (28) should be satisfied as well. Similarly, inequalities (29) and (30) should be satisfied whenever permanent deformation occurs and whenever such deformation is accompanied by damage, respectively.

This establishes the general thermodynamic framework. The decompositions (8) and (19) result in forms where the coupling between viscoelasticity and damage, and between permanent deformation and damage are split into two separate entities. This subject is discussed in more details in the following two sections.

3. VISCOELASTICITY COUPLED WITH DAMAGE

For the sake of generality and to encompass a majority of practical applications, the viscoelastic constitutive equations will be first formulated for a general case of nonlinear viscoelasticity coupled with damage. The resulting equations will subsequently be reduced to linear viscoelasticity coupled with damage.

3.1. General Formulation

The formulation will be first obtained for the case of fixed stress σ_{ij} , damage ω_{ijkl} and temperature T . The results will then be extended to fluctuating σ_{ij} , ω_{ijkl} and T .

For fixed σ_{ij} , ω_{ijkl} and T , an irreversible thermodynamic process is triggered in the material, which prompts the viscoelastic internal state variables γ_r to drift spontaneously toward their equilibrium values $\gamma_r^e = \gamma_r^e(\sigma_{ij}, \omega_{abcd}, T)$. These equilibrium values are assumed to be continuous and sufficiently differentiable functions of their arguments. Assuming that γ_r and γ_r^e are sufficiently small, a Taylor series expansion for ϕ^v about γ_r^e takes the form

$$\phi^v = \phi_e^v + \frac{1}{2} \phi_{rq}^v (\gamma_r - \gamma_r^e) (\gamma_q - \gamma_q^e) + \text{H.O.T.}, \quad (31)$$

where

$$\phi_e^v = \phi_e^v(\sigma_{ij}, \omega_{abcd}, T)$$

is the value of ϕ^v at equilibrium, and

$$\phi_{rq}^v = \left(\frac{\partial^2 \phi^v}{\partial \gamma_r \partial \gamma_q} \right)_e$$

is a symmetric matrix assumed to be constant. In the above relations, the subscript "e" implies that a quantity is calculated at $\gamma_r = \gamma_r^e \quad \forall r$. Note that at equilibrium ϕ^v is minimum (Callen, 1960), and hence

$$\left(\frac{\partial \phi^v}{\partial \gamma_r} \right)_e = 0,$$

and

$$\phi_{rq}^v \delta \gamma_r \delta \gamma_q > 0.$$

Consequently, there is no linear term in (31) and ϕ_{rq}^v is a positive definite matrix. In view of (22), it is clear that neglecting the higher order terms (H.O.T.) in (31) corresponds to assuming a linear relation between the thermodynamic forces Γ_r and the internal state variables γ_r . Such a linear relation is believed to be valid at least in the neighborhood of equilibrium (Prigogine, 1967). In Appendix A, we also show that neglecting the H.O.T. in (31) is equivalent to neglecting the same H.O.T. in the internal entropy production. It should be mentioned that an expansion similar to that in (31) was previously used by Lubliner (1972) in a strain formulation for viscoelasticity.

Employing the usual assumption of viscous-like resistance (Biot, 1954, 1955; Schapery, 1964, 1969), let

$$\Gamma_r = a_{rq} \dot{\gamma}_q, \quad (32)$$

where, according to Onsager's principle (Callen, 1960; Fung, 1965; deGroot and Mazur, 1984), $a_{rq} = a_{rq}(\sigma_{ij}, \omega_{abcd}, T)$ is a symmetric matrix. Substituting (32) into inequality (27)

$$a_{rq} \dot{\gamma}_r \dot{\gamma}_q \geq 0.$$

Hence, the matrix a_{rq} is positive semi-definite.

By analogy with Biot (1954, 1955) and Schapery (1964, 1969), we assume that all a_{rq} have common dependence on σ_{ij} , ω_{ijkl} and T ; namely

$$a_{rq} = a(\sigma_{ij}, \omega_{abcd}, T) \cdot a_{rq}^o, \quad (33)$$

where a is a positive scalar valued function of its arguments, and a_{rq}^o is a constant symmetric and positive semi-definite matrix. Equations (22), (31) and (33) yield

$$a_{rq}^o \frac{d\gamma_q}{d\xi} + \phi_{rq}^v \gamma_q = \phi_{rq}^v \gamma_q^e, \quad (34)$$

where ξ is a reduced time defined by $d\xi = dt/a$. Assuming that $\xi = 0$ at $t = 0$, then for fixed σ_{ijkl} , ω_{ijkl} and T one has $\xi = t/a$.

Since a_{rq}^o is a constant symmetric positive semi-definite matrix and ϕ_{rq}^v is a constant symmetric positive definite matrix, equation (34) can be rewritten in a decoupled form (Meirovitch, 1967) as

$$A_r^o \frac{d\hat{\gamma}_r}{d\xi} + \Phi_r^v \hat{\gamma}_r = \Phi_r^v \hat{\gamma}_r^e \quad (\text{no sum over } r), \quad (35)$$

where $\hat{\gamma}_r$ is a transformed set of internal state variables, each being a linear combination of the original internal state variables γ_q . The parameters $\hat{\gamma}_r^e$ are the equilibrium values corresponding to $\hat{\gamma}_r$ and are obtained from γ_q^e by the same linear transformation as that for $\hat{\gamma}_r$. Also, A_r^o and Φ_r^v are constants such that $A_r^o \geq 0$ and $\Phi_r^v > 0$.

For fixed stress, damage and temperature, the solution of equation (35) is

$$\hat{\gamma}_r = \hat{\gamma}_r^e \left(1 - e^{-\xi/\tau_r}\right) \quad (\text{no sum over } r), \quad (36)$$

where τ_r are retardation times given by

$$\tau_r = \frac{A_r^o}{\Phi_r^v} \quad (\text{no sum over } r). \quad (37)$$

In terms of the transformed internal state variables, expansion (31) is rewritten as

$$\phi^v = \phi_e^v + \frac{1}{2} \sum_r \Phi_r^v (\hat{\gamma}_r - \hat{\gamma}_r^e)^2 + \text{H.O.T.} \quad (38)$$

The viscoelastic strain can now be obtained by substituting (38) into (11) bearing in mind that γ_r , and hence $\hat{\gamma}_r$, are to be kept fixed during the partial differentiation indicated in (11). Employing (36) we then obtain

$$\varepsilon_{ij}^v = -\frac{\partial \phi_o}{\partial \sigma_{ij}} + \sum_r \left(1 - e^{-\xi/\tau_r}\right) \frac{\partial \Lambda_r}{\partial \sigma_{ij}}, \quad (39)$$

where

$$\Lambda_r = \frac{1}{2} \Phi_r^v (\hat{\gamma}_r^e)^2 \quad (\text{no sum over } r), \quad (40)$$

and

$$\phi_o = \phi_e^v + \sum_r \Lambda_r. \quad (41)$$

ϕ_o is the portion of the Gibb's free energy associated with instantaneous deformation.

Recall that equation (39) was obtained for fixed stress, damage and temperature. For fluctuating stress, damage and temperature the strain can be obtained by applying the superposition principle to expression (39). Following Pipkin (1986), ξ is identified as an intrinsic material time scale for the viscoelastic processes. If σ_{ij} , ω_{ijkl} and T can be expressed in terms of the material time ξ rather than physical time t , then $\partial \Lambda_r / \partial \sigma_{ij}$ becomes a function of ξ . Straightforward application of the superposition principle (Pipkin, 1986) yields

$$\varepsilon_{ij}^v = -\frac{\partial \phi_o}{\partial \sigma_{ij}} + \int_{0-}^{\xi} \sum_r \left[1 - e^{-(\xi-\xi')/\tau_r}\right] \frac{d}{d\xi'} \left(\frac{\partial \Lambda_r}{\partial \sigma_{ij}}\right) d\xi', \quad (42)$$

where

$$\xi = \int_0^t \frac{dt'}{a[\sigma_{ij}(t'), \omega_{abcd}(t'), T(t')]}, \quad (43)$$

and for $0 \leq \tau \leq t$

$$\xi' = \int_0^{\tau} \frac{dt'}{a[\sigma_{ij}(t'), \omega_{abcd}(t'), T(t')]}. \quad (44)$$

Noting that $d\xi = dt/a$ and $d\xi' = d\tau/a$ the variable of integration in (42) can be reverted form ξ' to τ . Hence

$$\varepsilon_{ij}^v = -\frac{\partial \phi_o}{\partial \sigma_{ij}} + \int_{0-}^t \sum_r \left[1 - e^{-(\xi-\xi')/\tau_r}\right] \frac{d}{d\tau} \left(\frac{\partial \Lambda_r}{\partial \sigma_{ij}}\right) d\tau, \quad (45)$$

Allowing for spatial variations of stress, damage and temperature, the total derivative $d/d\tau$ in (45) is replaced by a partial derivative, holding the spatial coordinates x_i fixed, so that

$$\varepsilon_{ij}^v = -\frac{\partial \phi_o}{\partial \sigma_{ij}} + \int_{0^-}^t \sum_r \left[1 - e^{-(\xi - \xi')/\tau_r} \right] \frac{\partial}{\partial \tau} \left(\frac{\partial \Lambda_r}{\partial \sigma_{ij}} \right) d\tau, \quad (46)$$

and from (43) and (44)

$$\xi - \xi' = \int_{\tau'}^t \frac{dt'}{a(\sigma_{ij}, \omega_{abcd}, T)}. \quad (47)$$

The first term on the right-hand side of (46) represents the instantaneous part of the strain, while the second term given by the integral represents the time-dependent portion. In Appendix B, it is shown that with some further simplifying assumptions the second term can be considered as a generalization of the well-known modified superposition integral (e.g. Schapery, 1969).

Using (22), (25), (31), (32) and the transformed internal state variables $\hat{\gamma}_r$, the dissipation inequality (28) can now be expressed as

$$\sum_r A_r^o \dot{\gamma}_r^2 - \left[\frac{\partial \phi_o}{\partial \omega_{ijkl}} - \sum_r \left(\frac{\hat{\gamma}_r}{\hat{\gamma}_r^e} \right) \frac{\partial \Lambda_r}{\partial \omega_{ijkl}} \right] \dot{\omega}_{ijkl} \geq 0. \quad (48)$$

The complete nonlinear formulation requires an expression for the evolution of the damage tensor ω_{ijkl} such that (48) is satisfied. Such expressions can be formally derived from thermodynamic considerations through the introduction of a damage potential (e.g. Lemaitre and Chaboche, 1985; Lemaitre, 1992), but their usefulness seems to be restricted to elastic response with damage. Alternatively, damage evolution relations may be associated with a damage function akin to a yield function in plasticity (Hansen and Schreyer, 1994). Another approach, which is guided by insights from viscoelastic fracture mechanics, was given by Schapery (1981, 1994, 1996). This approach suggests that an adequate accounting for damage evolution is obtained by incorporating damage effects only within the instantaneous portion of ε_{ij}^v . This assumption resulted in good predictions for epoxy resin composites. In practice, the evolution equations depend on the particular material considered, and several such expressions can be found in the literature (e.g. Krajcinovic, 1989). A form of these evolution equations will be considered in the following subsection.

3.2. Linear Viscoelasticity Coupled with Damage

Under sufficiently small stress levels, the viscoelastic behavior of polymeric composites is linear. In this section the foregoing general formulation is reduced to the case of a linear viscoelastic behavior coupled with damage. First, consideration is given to the general case of anisotropy with respect to both the undamaged material behavior and the ensuing damage response. Subsequently, attention is given to the simple case of isotropic material behavior that remains isotropic even after development of damage. In both cases, attention is restricted to a linear viscoelastic behavior where strain depends on current values of temperature but not on temperature history. Most amorphous polymers exhibit such behavior (Ferry, 1980). This latter restriction simplifies the formulation since it allows us to discard coupling between damage and temperature history. The extension to the general case of temperature-history dependent strain can be made following the same approach adopted here.

3.2.1. Generally anisotropic behavior

Consider first the time-dependent part of ε_{ij}^v . Since this part of strain is assumed to be independent of temperature history, it follows that all γ_r^e , and hence all $\hat{\gamma}_r^e$ and Λ_r , are independent of temperature. Consequently

$$\Lambda_r = \Lambda_r(\sigma_{ij}, \omega_{abcd}) .$$

Expanding Λ_r in terms of stress around a reference state ($\sigma_{ij} = 0, T = T_R$) up to quadratic terms to retain linearity one obtains

$$\Lambda_r = \frac{1}{2} \left(\frac{\partial^2 \Lambda_r}{\partial \sigma_{ij} \partial \sigma_{kl}} \right)_0 \sigma_{ij} \sigma_{kl} , \quad (49)$$

where the subscript 0 implies that a quantity is calculated at the reference state. Note that the constant term in (49) vanishes since $\gamma_r^e = 0$, and hence $\hat{\gamma}_r^e = \Lambda_r = 0$, at the reference state. In addition, the linear term in (49) is discarded since it corresponds to a residual strain at the reference state, which is disregarded in the present formulation.

A more specific functional form for Λ_r can be obtained by realizing that the internal molecular motions represented by γ_r occur on a much smaller dimensional scale than that of damage represented by ω_{ijkl} . This suggests that all γ_r^e , and hence all $\hat{\gamma}_r^e$ and Λ_r , are likely to

be affected by damage in a common manner; i.e. they have common dependence on ω_{abcd} . Consequently, we can write

$$\left(\frac{\partial^2 \Lambda_r}{\partial \sigma_{ij} \partial \sigma_{kl}} \right)_0 = \Delta S_{ijps}^r P_{pskl} \quad \forall r. \quad (50)$$

In (50), ΔS_{ijkl}^r is a symmetric fourth rank tensor ($\Delta S_{ijkl}^r = \Delta S_{klij}^r = \Delta S_{jikl}^r = \Delta S_{ijlk}^r$), and $P_{ijkl} = P_{ijkl}(\omega_{abcd})$ is a symmetric fourth rank tensor valued function of the damage variable ω_{abcd} such that

$$\text{at } \omega_{abcd} = 0 \quad \longrightarrow \quad P_{ijkl} = I_{ijkl}, \quad (51)$$

where

$$I_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

is the unit fourth rank tensor and δ_{ij} is Kronecker delta.

In view of the above hypothesis, it is also reasonable to assume that the intrinsic viscoelastic material time scale ξ is unaffected by damage. Moreover, for linear viscoelastic behavior, the effect of stress on the material time scale is negligible (Ferry, 1980). Thus, the scalar valued function a in (33) takes the form

$$a = a_T(T), \quad (52)$$

and represents the usual time-temperature shift factor in thermorheologically simple materials (Morland and Lee, 1960).

Employing (49), (50) and (52), the viscoelastic strain-stress relation (46) can be written as

$$\varepsilon_{ij}^v = -\frac{\partial \phi_o}{\partial \sigma_{ij}} + \int_{0-}^t \Delta S_{ijkl}(\xi - \xi') \frac{\partial \tilde{\sigma}_{kl}}{\partial \tau} d\tau, \quad (53)$$

where

$$\Delta S_{ijkl}(\xi) = \sum_r \Delta S_{ijkl}^r \left(1 - e^{-\xi/\tau_r}\right), \quad (54)$$

is the time-dependent compliance tensor of the undamaged (virgin) material,

$$\xi = \int_0^t \frac{dt'}{a[T(t')]} \quad , \quad \xi' = \int_0^t \frac{dt'}{a[T(t')]} \quad (55)$$

and

$$\tilde{\sigma}_{ij} = P_{ijkl} \sigma_{kl}. \quad (56)$$

Equation (53) suggests that the viscoelastic strain ε_{ij}^v is affected by damage indirectly through the stress $\tilde{\sigma}_{ij}$, which is the applied stress σ_{ij} mapped by the tensor P_{ijkl} . Following common practice in continuum damage mechanics, we designate $\tilde{\sigma}_{ij}$ as the effective stress.

Consider now the instantaneous part of ε_{ij}^v . The equilibrium viscoelastic Gibb's free energy ϕ_e^v in (38) can also be expanded around the reference state in the form

$$\phi_e^v = -\alpha_{ij}^o \sigma_{ij} \Delta T - \frac{1}{2} \left(\frac{\partial^2 \phi_e^v}{\partial \sigma_{ij} \partial \sigma_{kl}} \right)_0 \sigma_{ij} \sigma_{kl}, \quad (57)$$

where

$$\alpha_{ij}^o = \alpha_{ij}^o(\omega_{abcd}) = - \left(\frac{\partial^2 \phi_e^v}{\partial T \partial \sigma_{ij}} \right)_0,$$

is the thermal expansion tensor, $\Delta T = T - T_R$ is temperature excursion from its reference value T_R , and the subscript 0 has the same meaning as before. Following common practice in damage mechanics (e.g. Lemaitre and Chaboche 1985; Lemaitre, 1992), we further assume that $(\partial^2 \phi_e^v / \partial \sigma_{ij} \partial \sigma_{kl})_0$ depends on damage in the separable form

$$\left(\frac{\partial^2 \phi_e^v}{\partial \sigma_{ij} \partial \sigma_{kl}} \right)_0 = S_{ijps}^e Q_{pskl}, \quad (58)$$

where $Q_{ijkl} = Q_{ijkl}(\omega_{abcd})$ such that $Q_{ijkl} = I_{ijkl}$ at $\omega_{abcd} = 0$, and S_{ijkl}^e is the undamaged equilibrium compliance tensor. Using (41), the instantaneous strain takes the form

$$-\frac{\partial \phi_o}{\partial \sigma_{ij}} = \alpha_{ij}^o \Delta T + [S_{ijps}^e Q_{pskl} - \sum_r \Delta S_{ijps}^r P_{pskl}] \sigma_{kl}, \quad (59)$$

It is clear that, in general, the instantaneous part of strain can depend on damage in a manner that differs from that of the time-dependent part. However, for simplicity let us assume that both parts have the same dependence on damage so that

$$Q_{ijkl} = P_{ijkl}. \quad (60)$$

From (59), the instantaneous strain then becomes

$$-\frac{\partial \phi_o}{\partial \sigma_{ij}} = \alpha_{ij}^o \Delta T + S_{ijkl}^o \tilde{\sigma}_{kl}, \quad (61)$$

where

$$S_{ijps}^o = S_{ijkl}^e - \sum_r \Delta S_{ijkl}^r, \quad (62)$$

is the instantaneous (elastic) compliance tensor of the undamaged material.

The total strain given by (53) can now be rewritten in the compact form

$$\epsilon_{ij}^v = \alpha_{ij}^o \Delta T + \int_{0-}^t S_{ijkl}(\xi - \xi') \frac{\partial \tilde{\sigma}_{kl}}{\partial \tau} d\tau , \quad (63)$$

where

$$S_{ijkl}(\xi) = S_{ijkl}^o + \Delta S_{ijkl}(\xi) , \quad (64)$$

is the overall (instantaneous and time-dependent) compliance tensor of the undamaged material. It is well-known (e.g. Fung, 1965) that both S_{ijkl}^o and S_{ijkl}^e are positive definite. From (62) it can then be hypothesized that each ΔS_{ijkl}^e is positive semi-definite, and from (54) it follows that $\Delta S_{ijkl}(\xi)$ is also positive semi-definite, as it should be.

In the remainder of this work we consider isothermal conditions, namely $\Delta T = 0$. The constitutive equation (63), then reads

$$\epsilon_{ij}^v = \int_{0-}^t S_{ijkl}(t - \tau) \frac{\partial \tilde{\sigma}_{kl}}{\partial \tau} d\tau . \quad (65)$$

Also, the dissipation inequality (48) becomes

$$\sum_r A_r^o \dot{\gamma}_r^2 + S_{ijps}(t) \dot{P}_{pskl} \sigma_{ij} \sigma_{kl} \geq 0 . \quad (66)$$

Since the first term on the left-hand side of (66) is always non-negative, a sufficient condition to satisfy (66) is

$$S_{ijps}(t) \dot{P}_{pskl} \longrightarrow \text{positive semi-definite} , \quad (67)$$

where

$$\dot{P}_{pskl} = \frac{\partial P_{pskl}}{\partial \omega_{abcd}} \dot{\omega}_{abcd} .$$

Thus, the functional form of the mapping tensor P_{ijkl} is restricted by the two requirements in (51) and (67). One such possible functional form (e.g. Ju, 1990) is

$$P_{ijkl} = (I_{ijkl} - \omega_{ijkl})^{-1} , \quad (68)$$

which is considered to be a generalization to the scalar form used by Kachanov (1958) in one-dimensional problems. Clearly, (68) satisfies (51), and the choice of ω_{ijkl} should be such that (67) is satisfied.

3.2.2. Isotropic behavior

For an isotropic undamaged material behavior, the instantaneous compliance in (62) takes the form (e.g. Fung, 1965)

$$S_{ijkl}^o = \frac{1}{2} J_o I_{ijkl} + \frac{1}{9} \left(B_o - \frac{3}{2} J_o \right) \delta_{ij} \delta_{kl}, \quad (69)$$

where J_o and B_o are the instantaneous shear and bulk compliances, respectively. Similarly, the time-dependent compliance in (54) takes the form

$$\Delta S_{ijkl}(t) = \frac{1}{2} \Delta J(t) I_{ijkl} + \frac{1}{9} \left[\Delta B(t) - \frac{3}{2} \Delta J(t) \right] \delta_{ij} \delta_{kl}, \quad (70)$$

where, analogously

$$\Delta J(t) = \sum_r \Delta J_r \left(1 - e^{-t/\tau_r} \right), \quad (71)$$

and

$$\Delta B(t) = \sum_r \Delta B_r \left(1 - e^{-t/\tau_r} \right), \quad (72)$$

with ΔJ_r and ΔB_r positive constants. The overall compliance can now be expressed in the form

$$S_{ijkl}(t) = \frac{1}{2} J(t) I_{ijkl} + \frac{1}{9} \left[B(t) - \frac{3}{2} J(t) \right] \delta_{ij} \delta_{kl}, \quad (73)$$

where

$$J(t) = J_o + \Delta J(t),$$

and

$$B(t) = B_o + \Delta B(t),$$

are the shear and bulk compliances, respectively.

Consider the circumstance that the response of the damaged material remains isotropic. In this case, damage must admit an isotropic representation that does not change the original material symmetry properties. One possible way of representing isotropic damage is to introduce two independent scalar variables ω_S and ω_H that characterize the influence of damage on the deviatoric and hydrostatic strain components, respectively. The effective stress $\tilde{\sigma}_{ij}$ is then assumed in the form

$$\tilde{\sigma}_{ij} = \frac{\sigma'_{ij}}{(1 - \omega_S)} + \frac{1}{3} \frac{\sigma_{kk}}{(1 - \omega_H)} \delta_{ij}, \quad (74)$$

where

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

is the deviatoric part of stress and σ_{kk} is the trace of the stress tensor. Also, in (74) both ω_S and ω_H are in general functions of a scalar ω representing the microcrack density, referred to as the microcrack concentration parameter (Kachanov, 1986).

The mapping tensor P_{ijkl} can be evaluated from (56) and (74), to read

$$P_{ijkl} = \frac{1}{(1 - \omega_S)} I_{ijkl} + \frac{(\omega_H - \omega_S)}{3(1 - \omega_S)(1 - \omega_H)} \delta_{ij} \delta_{kl}. \quad (75)$$

Substitution of this result into (68) gives

$$\omega_{ijkl} = \omega_S I_{ijkl} + \frac{1}{3} (\omega_H - \omega_S) \delta_{ij} \delta_{kl}. \quad (76)$$

Equation (76) takes the most general form of a fourth rank isotropic damage tensor ω_{ijkl} (Ju, 1990). Upon substitution of (73) and (74) into (65), the isothermal strain-stress relation of an isotropic damaged viscoelastic response takes the form

$$\varepsilon_{ij}^v = \int_{0-}^t \left[\frac{1}{2} J(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma'_{ij}}{1 - \omega_S} \right) + \frac{1}{9} B(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma_{kk}}{1 - \omega_H} \right) \delta_{ij} \right] d\tau. \quad (77)$$

The deviatoric and hydrostatic parts of the material behavior can be separated from (77) into the following respective forms

$$2 \varepsilon_{ij}^{v'} = \int_{0-}^t J(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma'_{ij}}{1 - \omega_S} \right) d\tau, \quad (78)$$

and

$$\varepsilon_{kk}^v = \int_{0-}^t B(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma_{kk}/3}{1 - \omega_H} \right) \delta_{ij} d\tau, \quad (79)$$

where $\varepsilon_{ij}^{v'}$ and ε_{kk}^v are the deviatoric and hydrostatic strain components, respectively. The effects of ω_S and ω_H on material behavior may then be evaluated from uniaxial creep tests. For instance, let $\sigma_{ij} = \sigma \delta_{i1} \delta_{j1}$, in which case $\varepsilon_{22}^v = \varepsilon_{33}^v$, then (78) and (79) yield

$$\varepsilon_{11}^v - \varepsilon_{22}^v = \int_{0-}^t \frac{1}{2} J(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma}{1 - \omega_S} \right) d\tau, \quad (80)$$

and

$$\varepsilon_{11}^v + 2 \varepsilon_{22}^v = \int_{0-}^t \frac{1}{3} B(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\sigma}{1 - \omega_H} \right) d\tau. \quad (81)$$

Obviously, equations (80) and (81) predict a stress and time-dependent Poisson's ratio, $\nu = \nu(\sigma, t)$. In principle, these equations can be resolved from measurements of longitudinal and transverse strains.

Isotropic damage thus can be represented by the two scalar functions ω_S and ω_H . These functions are subjected to two restrictions. Firstly, recalling equations (51), (68) and (76), they must vanish in the absence of damage. Secondly, they must satisfy the requirement in (67). Using (73) for $S_{ijkl}(t)$ and (75) for P_{ijkl} , the left-hand side of (67) becomes

$$S_{ijps}(t) \dot{P}_{pskl} = \frac{\dot{\omega}_S}{(1 - \omega_S)^2} \frac{J(t)}{2} \left(I_{ijkl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) + \frac{\dot{\omega}_H}{(1 - \omega_H)^2} \frac{B(t)}{9} \delta_{ij} \delta_{kl}. \quad (82)$$

For the above expression to be positive semi-definite, it suffices that both $\dot{\omega}_S$ and $\dot{\omega}_H$ be non-negative. This implies that damage can only increase or cease to increase but it cannot decrease (i.e. no microcrack healing). This situation encompasses a majority of practical applications involving damage.

To complete the isotropic formulation, evolution equations for ω_S and ω_H are needed. Under creep loadings with increasing damage, such equations may be taken in the well-known Kachanov-Rabotnov form (Kachanov, 1986)

$$\dot{\omega}_S = \left[\frac{\langle \sigma_e^S - \sigma_c^S \rangle}{A_S (1 - \omega_S)} \right]^{\chi_S}, \quad (83)$$

and

$$\dot{\omega}_H = \left[\frac{\langle \sigma_e^H - \sigma_c^H \rangle}{A_H (1 - \omega_H)} \right]^{\chi_H}. \quad (84)$$

In equations (83) and (84), σ_e^S and σ_e^H are equivalent stresses for the deviatoric and hydrostatic damage modes, respectively, and σ_c^S and σ_c^H , considered to be material properties, are the corresponding threshold stresses below which no damage develops. Also, χ_S , χ_H , A_S , and A_H are material properties characteristic of creep damage, which are typically strongly dependent on temperature. Finally, $\langle \cdot \rangle$ denotes the McAuley bracket, i.e. $\langle x \rangle = x$ if $x > 0$ and $\langle x \rangle = 0$ if $x \leq 0$. Guided by Leckie and Hayhurst (1977), the equivalent stresses σ_e^S and σ_e^H may be taken in the following forms

$$\sigma_e^S = \sqrt{\frac{3}{2}} \sigma'_{ij} \sigma'_{ij}, \quad (85)$$

and

$$\sigma_e^H = \beta \sigma_1 + (1 - \beta) \sigma_{kk}, \quad (86)$$

where σ_1 is the maximum principal tensile stress and β ($0 < \beta < 1$) is a material property.

For monotonic creep loadings the stresses σ_c^S and σ_c^H serve as thresholds for the onset of damage. For more complex loading histories involving, for instance, complete or partial stress removals, the onset of damage is usually not related to a specific threshold stress. In this case, the maximum strain achieved during the loading history may serve as a better indication for the onset of new damage (Gurtin and Francis, 1981; Suvorova, 1985). For instance, after a complete or a partial load removal damage evolution usually ceases. If afterwards the previously achieved maximum strain is reached by increasing the applied load or by creep under a lower load, then damage would start to evolve again.

Finally, it should be mentioned that a simpler representation, where isotropic response is retained even after the onset of damage, can be obtained as a special case of (74) by setting $\omega_H = \omega_S = \omega$. Consequently, damage is represented by the scalar ω so that $\omega_{ijkl} = \omega I_{ijkl}$ and the effective stress becomes

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - \omega}, \quad (87)$$

This effective stress coincides with the original form proposed by Kachanov (1958). Under creep loadings, the evolution equation for ω can again be taken in the foregoing Kachanov-Rabotnov form (Leckie and Hayhurst, 1977). Note that in contrast with the elastic case, where scalar damage corresponds to a constant Poisson's ratio (Ju, 1990), in the viscoelastic case Poisson's ratio remains both time and stress dependent.

4. PERMANENT DEFORMATION

The viscoelastic expressions derived in the foregoing section predict strains that recover completely after load removal and cannot account for permanent deformations that may persist indefinitely. As mentioned in Section 1, experimental evidence suggests that these permanent strains increase with load during monotonic loading as well as with creep duration. In other words, permanent strains in polymeric composites may depend on both stress magnitude and history. The foregoing permanent deformation can be attributed to several deformation mechanisms, which may require disparate modeling approaches. In the remainder of this section we do not prescribe a specific analytical model for the permanent deformation but provide a brief review of some of the possible modeling approaches. The

thermodynamics framework formulated in Section 2 is sufficiently general to accommodate any one or a combination of these modeling methods.

Since permanent strains in polymeric composites may depend, in general, on both stress and stress history, ε_{ij}^p in (8) should be further decomposed into a time-independent component ε_{ij}^{p0} and a time-dependent component ε_{ij}^{pt} , namely

$$\varepsilon_{ij}^p = \varepsilon_{ij}^{p0} + \varepsilon_{ij}^{pt}. \quad (88)$$

In principle, these components can be modeled, respectively, following concepts in phenomenological plasticity and viscoplasticity (e.g. Sun and Chen, 1989; Ha *et al.*, 1991). For instance, experimental investigations (Smith and Weitsman, 1996) reveal that under uniaxial tensile creep loadings, the uniaxial time-dependent component ε^{pt} may be expressed in the form

$$\varepsilon^{pt} = B (\sigma - \sigma^*)^\kappa t^\mu, \quad (89)$$

where σ^* is a threshold stress below which no permanent deformation takes place, t is time, and B, μ and κ are constants. It should be mentioned that an expression similar to that in (89) was also employed by Tuttle *et al.* (1995) for the time-dependent permanent deformation of a polymeric composite. Differentiating (89) with respect to time and eliminating t in favor of ε^{pt} and $(\sigma - \sigma^*)$ we obtain

$$\sigma - \sigma^* = \left(\frac{1}{\mu B^{1/\mu}} \right)^{\mu/\kappa} (\varepsilon^{pt})^{(\mu-1)/\kappa} (\dot{\varepsilon}^{pt})^{\mu/\kappa}. \quad (90)$$

This stress-strain relation is in the well-known power-law form of viscoplasticity (Lubliner, 1990), thereby suggesting that ε^{pt} may be modeled by viscoplasticity theory. A similar approach can be taken to model ε^{p0} using time-independent plasticity instead of viscoplasticity.

An approach to model plastic deformation in viscoelastic solids has also been proposed by Naghdi and Murch (1963). This modeling effort can predict the onset of plastic deformation under general circumstances and implicitly accounts for the decomposition of ε^p in (88). It is important to realize that adopting phenomenological plasticity and viscoplasticity in modeling the permanent deformation associates the internal state variables ζ_n in Section 2 with internal stresses that represent isotropic and kinematic hardening. In this case, coupling between permanent deformation and damage can be modeled following, for instance, approaches by Ju (1989) and Hansen and Schreyer (1994).

Physically based approaches to model permanent deformation in polymeric materials have also been proposed (e.g. Lee *et al.*, 1993; Hasan and Boyce, 1995). These two studies in particular focus on semi-crystalline and glassy polymers, respectively, and hence could be suitable for polymeric composites with thermoplastic matrices. Also, in these modeling approaches it is assumed that the permanent deformation is associated with the ability of the polymeric matrix to undergo irreversible structural changes which, correspondingly, leads to non-recoverable strains. In polymeric composites, however, permanent deformation can also arise from other mechanisms such as frictional sliding following fiber-matrix interfacial microdebondings. A mechanistic approach for modeling such phenomena exists for unidirectional fibers embedded in an elastic matrix (Hutchinson and Jensen, 1990). No similar models exist for a more complex fiber architecture (e.g. randomly oriented fibers) or more complex matrix behavior (e.g. viscoelastic matrix).

Finally, permanent deformation can also arise following microcracking. This is attributed to the formation of microcrack surface roughness and other irregularities, due to fiber rotations and breakage, that resist microcrack closure upon unloading (e.g. Schapery, 1989; Smith and Weitsman, 1996). In this case, the internal state variables ζ_n in Section 2 can be associated with the internal surface roughness and the internal degrees of freedom due to fiber rotations. An approach similar to phenomenological plasticity can also be adopted to model the permanent deformation that result due to microcrack damage (Dragon and Mróz, 1979).

5. APPLICATION TO SWIRL-MAT POLYMERIC COMPOSITES

In this section, the foregoing constitutive model will be utilized to predict the response of a swirl-mat polymeric composite. The objective is to predict the creep-damage behavior of the material and provide guidelines for its life-time assessment procedures.

The material considered consists of an E-glass fiber preform embedded in a urethane matrix. A detailed description of the material along with its mechanical response is given by Smith and Weitsman (1996). Here, we only observe that under sufficiently low stress levels the material exhibits linear viscoelastic behavior, while under applied uniaxial tensile stresses that exceed a threshold level of $\sigma_c \approx 36$ MPa, damage in the form of profuse microcracks

is observed. Also, the relative magnitudes of the permanent strains attained during creep tests do not exceed 5% of the total strain. Consequently, from the practical point of view, it is justified to neglect the permanent deformation for the present purposes; i.e. $\varepsilon_{ij} \approx \varepsilon_{ij}^y$.

Experimental data for the material under consideration are available for uniaxial tensile loadings. Assuming scalar damage, the uniaxial form of the model in (65) and (87) reads

$$\varepsilon = \int_{0-}^t D(t-\tau) \frac{d}{d\tau} \left(\frac{\sigma}{1-\omega} \right) d\tau, \quad (91)$$

where $D(t)$ is the uniaxial compliance of the undamaged material. Similarly to (64), $D(t)$ can be decomposed into instantaneous $D_o = S_{1111}^o$ and time-dependent $\Delta D(t) = \Delta S_{1111}(t)$ parts, namely

$$D(t) = D_o + \Delta D(t), \quad (92)$$

where, in view of (54),

$$\Delta D(t) = \sum_r \Delta S_{1111}^r (1 - e^{-t/\tau_r}). \quad (93)$$

Upon replacing the sum of the discrete spectrum of retardation times by an integral of a continuous spectrum, it can be shown (Pipkin, 1986) that $\Delta D(t)$ may be expressed as

$$\Delta D(t) = D_1 t^n, \quad (94)$$

where D_1 and n ($0 < n < 1$) are constants, which can be readily determined from uniaxial creep tests under sufficiently low stress levels (i.e. without damage). For the material at hand, Smith and Weitsman (1996) determined $D_o = 9.6 \times 10^{-5} \text{ MPa}^{-1}$, $n = 0.08$, and $D_1 = 2.8 \times 10^{-5} \text{ MPa}^{-1}/\text{hour}^{0.08}$. A variability of about 20% is observed in the experimentally determined values for D_o . This variability is attributed to randomness in the fiber architecture in the swirl-mat as well as to manufacturing induced inhomogeneity. The abovementioned value for D_o is therefore taken as the average of the experimentally recorded values.

Similarly to (83) and (84), the evolution equation for ω is taken in the form

$$\dot{\omega} = \left[\frac{(\sigma - \sigma_c)}{A(1-\omega)} \right]^\chi, \quad (95)$$

where A and χ are characteristic creep damage properties. It should be noted that in creep tests, the stress is typically ramped over a short period of time up to the prescribed level. During the ramp loading stage the behavior is essentially linear elastic until the stress exceeds

the threshold value σ_c when damage starts to occur. Thus, for tests with $\sigma > \sigma_c$, creep occurs under an initial value of damage ω_o that accumulates during the load ramping stage and depends on the stress level σ . For creep tests with $\sigma > \sigma_c$, solution of (95) using the initial condition $\omega = \omega_o$ at $t = 0$ reads

$$\frac{1}{1 - \omega} = \frac{1}{1 - \omega_o} \left(1 - \frac{t}{t_c}\right)^{-1/(1+\chi)}, \quad (96)$$

where t_c is a normalizing constant given by

$$t_c = \frac{(1 - \omega_o)^{1+\chi}}{1 + \chi} \left(\frac{A}{\sigma - \sigma_c}\right)^\chi. \quad (97)$$

Denoting the value of ω corresponding to a complete failure of the material by ω_f , then the time to failure t_f can be obtained from (96) in the form

$$t_f = \left[1 - \left(\frac{1 - \omega_f}{1 - \omega_o}\right)^{1+\chi}\right] t_c. \quad (98)$$

From (91) and (96) it follows that a theoretical value for ω at complete failure is $\omega_c = 1$ and occurs at $t = t_c$. In practice, however, materials typically fail at values of $\omega_f < 1$ (e.g. Lemaitre and Chaboche, 1985, p. 364) corresponding to $t_f < t_c$.

To complete the uniaxial characterization of the material, values for ω_o , ω_f , χ and A are needed. If ω_o is known then creep-rupture tests can be performed at different stress levels to experimentally determine t_f as a function of stress, and subsequently determine the values of ω_f , χ and A .

Focusing attention on the instantaneous damage ω_o that occurs during the rapid ramp loading to the stress level σ , we note that in the case of elastic materials undergoing damage, the ratio of the unloading compliance to the loading compliance provides a measure for the level of damage. In particular, it is well-known (Kachanov, 1986) that

$$\frac{1}{1 - \omega_o} = \frac{D_o^u}{D_o}, \quad (99)$$

where D_o^u is the unloading compliance. To determine ω_o “spike” tests, each consisting of a constant stress rate loading-unloading cycle, were performed up to the stress levels $\sigma = 55, 69$, and 83 MPa corresponding, respectively, to approximately 40, 50, and 60% of the ultimate tensile strength. The rate of loading used was $\dot{\sigma} \approx 5.6$ MPa/s, which is sufficiently high to keep the viscoelastic effects at a minimal level. From these tests, values for D_o^u were determined. Values for D_o were also determined to eliminate the effect of sample to sample

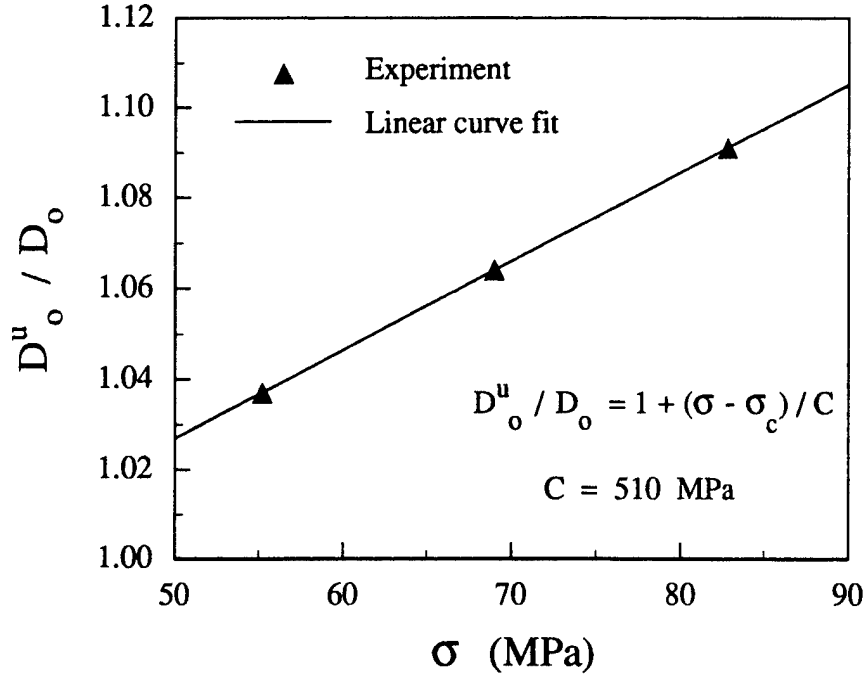


Figure 1: Ratio between the uniaxial unloading and loading compliances as a function of stress

variability in D_o . Figure 1, which shows the results of these three tests, clearly suggests a linear correlation between D_o^u / D_o and σ . Since for $\sigma < \sigma_c$ no damage occurs, then using (99) it is possible to write

$$\frac{1}{1 - \omega_o} = 1 + \frac{\langle \sigma - \sigma_c \rangle}{C}, \quad (100)$$

where C is a normalizing constant determined from Figure 1 as $C = 510$ MPa.

With the above expression for ω_o at hand, it is now possible to evaluate ω_f , A , and χ from time-to-failure data. Preliminary creep-rupture tests at temperature $T = 75^\circ\text{F}$ and 50% relative humidity (Corum, 1995) are shown in Figure 2. Clearly, these data exhibit a large amount of scatter which, again, is attributable to inhomogeneity and randomness of the swirl-mat polymeric composite material. Nevertheless, the data in Figure 2 can be used to provide some life-time estimates for the material. The constants ω_f , A , and χ were determined by fitting the experimental data to the expression for t_f in (98) using (100) for ω_o . Based on the best fit curve, depicted in Figure 2 by the solid line, the following values were determined

$$\omega_f = 0.67, \quad \chi = 7.1, \quad A = 260 \text{ MPa} \cdot \text{hour}^{1/7.1}. \quad (101)$$

In view of the large scatter in the creep-rupture data, the above values should be considered

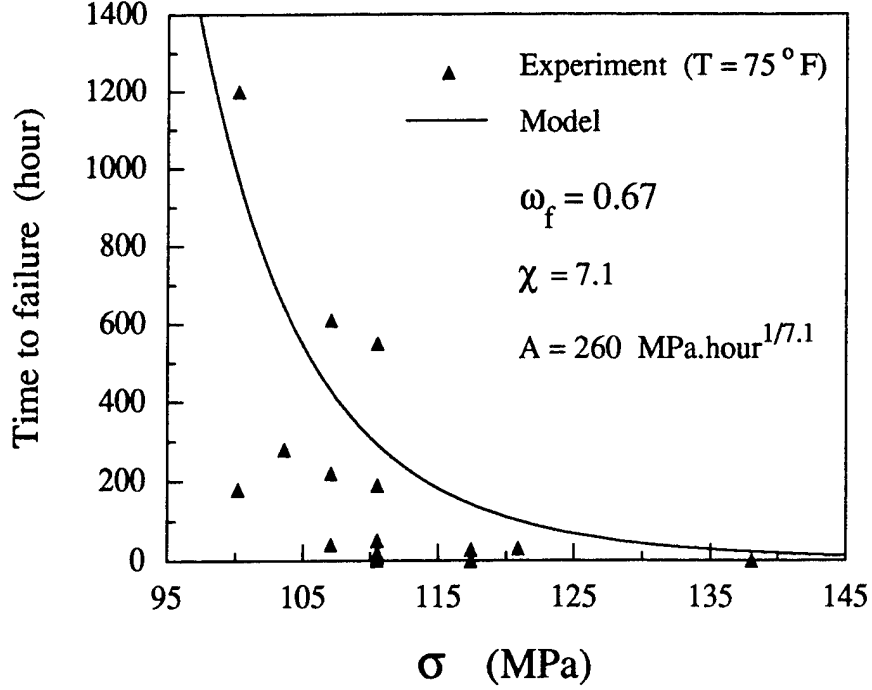


Figure 2: Time to failure as a function of stress in uniaxial creep-rupture tests

as preliminary estimates rather than conclusive material properties.

In addition to the aforementioned inherent variability in properties of the swirl-mat composite, a factor that significantly contributes to the scatter in creep-rupture data is the high sensitivity of the behavior of the considered material to fluctuations in the ambient environment (temperature and relative humidity). Creep-rupture tests typically require long durations, where uncontrollable fluctuations in the environment accumulate and can affect the results significantly. It is apparent that the large scatter in the limited experimental data in Figure 2 undermines the reliability of predictions based upon the values listed in (101). We therefore restrict attention only to times that are short in comparison with the time to failure t_f , so that the effect of environmental fluctuations can be neglected.

Short term creep tests (approximately 170 hours) were performed under stress levels of 55, 69, and 83 MPa at temperature $T = 75^\circ\text{F}$ and 50% relative humidity. The above stresses were ramped at the same rate as that for the tests in Figure 1, so that the expression for ω_o in (100) remains applicable. For creep under constant stress σ , substitution of (92), (94) and (96) into (91) yields

$$\varepsilon = D_o \frac{\sigma}{(1 - \omega)} + D_1 \frac{\sigma}{(1 - \omega_o)} \int_0^t (t - \tau)^n \frac{d}{d\tau} \left(1 - \frac{\tau}{t_c}\right)^{-1/(1+\chi)} d\tau. \quad (102)$$

Upon integrating by parts and changing the integration variable so that

$$\eta = \tau/t,$$

equation (102) becomes

$$\varepsilon = D_o \frac{\sigma}{(1-\omega)} + n D_1 t^n \frac{\sigma}{(1-\omega_o)} \int_0^1 (1-\eta)^{n-1} \left(1 - \frac{t}{t_c} \eta\right)^{-1/(1+\chi)} d\eta, \quad (103)$$

The solution of the integral in (103) can be obtained in a closed form (Gradsteyn and Ryzhik, 1965, Sec. 3.211, p. 287 and Sec. 9.182-10, p. 1055), so that

$$\varepsilon(t) = D_o \frac{\sigma}{(1-\omega)} + D_1 t^n \frac{\sigma}{(1-\omega_o)} F(1, \rho, 1+n; t/t_c), \quad (104)$$

where $F(1, \rho, 1+n; t/t_c)$ is a hypergeometric series that converges for all $t < t_c$ and is given by

$$\begin{aligned} F(1, \rho, 1+n; t/t_c) = 1 + \frac{\rho}{(n+1)} \left(\frac{t}{t_c}\right) + \frac{\rho(\rho+1)}{(n+1)(n+2)} \left(\frac{t}{t_c}\right)^2 \\ + \frac{\rho(\rho+1)(\rho+2)}{(n+1)(n+2)(n+3)} \left(\frac{t}{t_c}\right)^3 + \dots, \end{aligned}$$

and ρ is given by

$$\rho = \frac{1}{1+\chi}.$$

Experimental data along with model predictions are given in Figure 3. The experimental data are depicted by the scattered symbols while model predictions are represented by the solid lines. Note the good agreement between model predictions and experimental results.

Using (98), (100) and the values for the creep-rupture constants in (101), the times to failure are estimated as $t_f \approx 1.07 \times 10^7, 1.72 \times 10^5$ and 1.14×10^4 hours for the stress levels 55, 69, and 83 MPa, respectively. Obviously a creep duration of 170 hours represents only a very small fraction of the life time, during which the damage ω remains essentially constant at its initial value ω_o , as can be verified from (96). This is in agreement with the experimental observation (Smith and Weitsman, 1996) that after creep periods ranging from 0.5 to 170 hours the unloading compliance D_o^u remains essentially unchanged, implying that damage remains practically constant. Thus for short-time creep-damage behavior a good approximation to the constitutive equation (91) with constant σ is

$$\varepsilon \approx \frac{1}{1-\omega_o} D(t) \sigma. \quad (105)$$

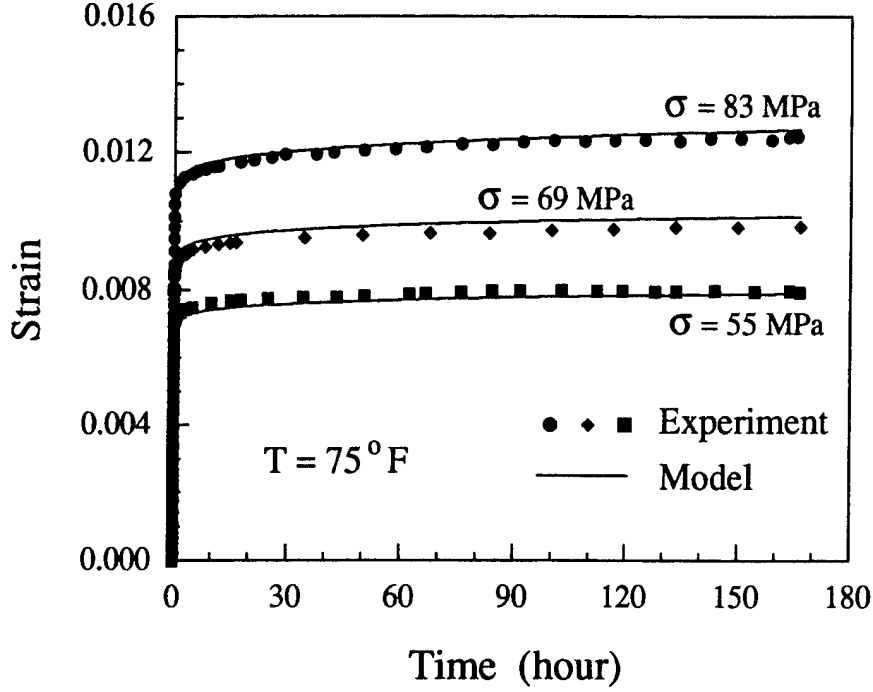


Figure 3: Comparison between model and experiment for short-term creep

Indeed, using (100), the difference between this approximation and exact calculation of the integral in (91) is quite negligible for the creep durations shown in Figure 3.

For illustration purposes, model prediction for long-term creep behavior up to the failure time t_f , as obtained from (91), for $\sigma = 83$ MPa is shown in Figure 4. This figure demonstrates that, similarly to metal, the material at hand exhibits a significant amount of tertiary creep prior to failure. Features of such behavior were observed in creep-rupture tests, especially at high temperatures (Corum, 1995). The fact that tertiary creep can be significant for the material considered herein is also consistent with the relatively large value found for ω_f .

6. CONCLUDING REMARKS

In this article a thermodynamically consistent framework is proposed for modeling different types of deformation of polymeric composites. The framework accommodates viscoelasticity, damage in the form of distributed microcracks, and both time independent and dependent permanent deformation. Several tensorial ranks for the damage variable can be used in the context of the present formulation. In this article, however, damage is represented by a

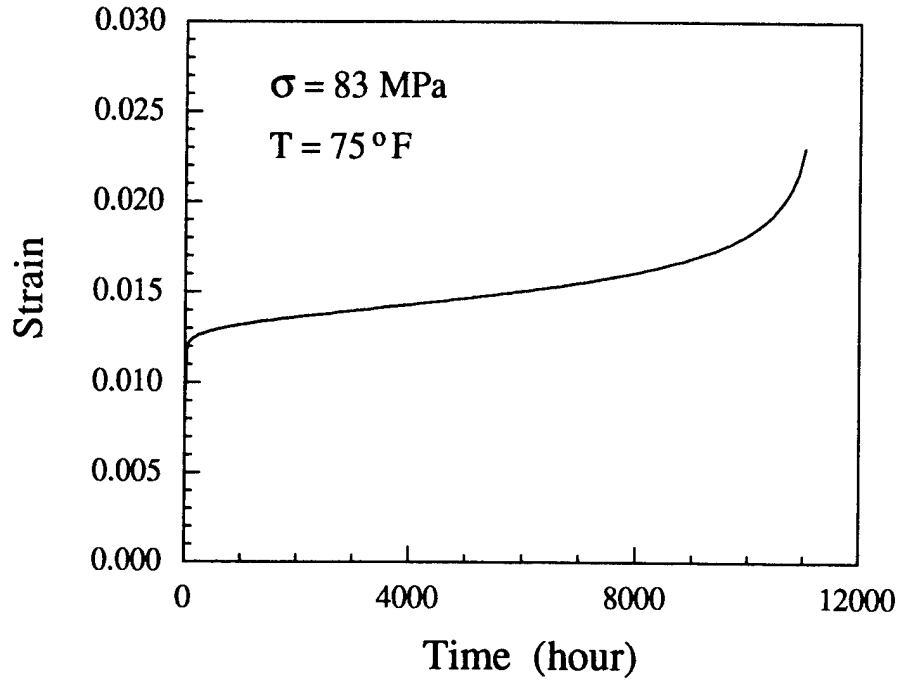


Figure 4: Model predictions for creep-damage behavior up to failure

general fourth rank tensor. Such a representation is capable of simulating general changes in material symmetry properties due to microcracking-induced damage.

Upon establishing the thermodynamics framework, a comprehensive approach was proposed for modeling the coupling between viscoelasticity and damage. The formulation was first established for a nonlinear viscoelastic behavior coupled with damage. It is shown in Appendix B that, in the absence of damage, the well-known modified superposition integral is a special case of the present formulation. The formulation is subsequently reduced to linear viscoelastic behavior coupled with damage. In this case, the effect of damage can be incorporated into the constitutive equations in a form consistent with the well-known effective stress concept. For linear viscoelastic behavior coupled with damage, two special cases of the generally anisotropic damage behavior were considered. In the first of these two cases damage is represented by two scalar functions ω_S and ω_H and was referred to as isotropic damage. In the second case damage was represented by a single scalar and was referred to as scalar damage.

Using scalar damage, a uniaxial form of the model for linear viscoelasticity coupled with damage was applied to a swirl-mat polymeric composite. When the deformation model was complemented by a Kachanov-Rabotnov form of damage evolution, an experimentally

observed tertiary creep stage could be predicted by the current formulation. This affords a convenient means for life-time assessment of structural components. It is noted that the estimated strain to failure of about 2.3% (Figure 4) is consistent with experimental data. This justifies the small strain formulation adopted in this work which, for the case of swirl-mat polymeric composites, remains valid up to failure.

It is interesting to note that a special sub-case of the present formulation has certain similarities with a somewhat reduced form of Schapery's non-linear viscoelastic model (Schapery, 1969). To show this, consider the general strain-stress relation given by (46) and (47). Adopting the functional forms in (49), (50), (57) and (58), and allowing for $Q_{ijkl} \neq P_{ijkl}$, then the strain-stress relation (53) remains valid with the instantaneous strain given by (59). For simplicity, consider isothermal conditions $\Delta T = 0$. Restricting attention to scalar damage and assuming that the material time scale ξ is unaffected by damage, then similarly to (91) the viscoelastic uniaxial strain-stress relation can be written as

$$\epsilon^v = f_o D_o \sigma + \int_{0-}^t \Delta D(\xi - \xi') \frac{d}{d\tau} (f_2 \sigma) d\tau, \quad (106)$$

where f_o and f_2 are scalar functions of the damage variable ω , and

$$\xi = \int_0^t \frac{dt'}{a[\sigma(t')]} \quad , \quad \xi' = \int_0^\tau \frac{dt'}{a[\sigma(t')]} .$$

If the tensor P_{ijkl} is taken in the form given by (68), then $f_2 = 1/(1 - \omega)$. However, since $Q_{ijkl} \neq P_{ijkl}$ in (106), f_o and f_2 can be disparate functions of ω . Since the evolution equation for ω can be written in a general form as $\dot{\omega} = \dot{\omega}(\sigma, \omega)$, then ω can in principle be expressed as a function of σ . Consequently, f_o and f_2 can be considered as compounded functions of σ and t .

Schapery's model (1969), for isothermal conditions and applied uniaxial stress, reads

$$\epsilon = g_o D_o \sigma + g_1 \int_{0-}^t \Delta D(\xi - \xi') \frac{d}{d\tau} (g_2 \sigma) d\tau, \quad (107)$$

where g_o, g_1 and g_2 are scalar functions of stress (but not of time). Comparison between (106) and (107) shows that the two forms are similar with the exception that in (106) the counterpart of g_1 can be thought of as a constant lumped within f_2 . It is worth noting that in the present formulation the physical origin of the scalar functions f_o and f_2 stems from damage. By contrast, in Schapery's formulation the functions g_o, g_1 and g_2 arise from a formal thermodynamic formulation. It should also be noted that unlike Schapery's formulation, in

the current work the Gibb's free energy function is expanded about the equilibrium values γ_r^e (equation 31), which leads directly to expression (50) for a damage modified compliance familiar in the context of continuum damage mechanics.

While emphasis in this work was placed on the viscoelastic part of the deformation, several deformation mechanisms and modeling approaches to the permanent deformation were also discussed and can be readily accommodated within the proposed thermodynamics framework. For simple loading cases, e.g. monotonic and creep loadings, applied to the swirl-mat polymeric composite considered in Section 5, experimental observations indicate that permanent strains are significantly lower than their viscoelastic counterparts and, hence, permanent deformation can justifiably be neglected. It should, however, be noted that under cyclic loading conditions, and for a sufficiently large number of cycles, permanent deformation may play an important role in the overall deformation.

APPENDIX A

In this appendix it is shown that neglecting the higher order terms (H.O.T.) in (31) is equivalent to neglecting the same H.O.T. in the internal entropy production. To this end, and to make the point in the simplest possible way, consider the circumstance where no permanent deformation occurs. Thus $\phi^v = \phi$, $\psi^v = \psi$ and we have (e.g. Lubliner, 1972)

$$\phi^v = u - \sigma_{ij} \varepsilon_{ij} - S T, \quad (\text{A.1})$$

where u is the internal energy per unit volume. To further simplify matters, consider the case of fixed stress, damage and temperature, then at equilibrium

$$\phi_e^v = u_e - \sigma_{ij} (\varepsilon_{ij})_e - S_e T, \quad (\text{A.2})$$

where, again, the subscript "e" implies that a quantity is evaluated at $\gamma_r = \gamma_r^e$. Subtracting (A.2) from (A.1) one gets

$$\phi^v - \phi_e^v = u - u_e - \sigma_{ij} [\varepsilon_{ij} - (\varepsilon_{ij})_e] - T \Delta S, \quad (\text{A.3})$$

where

$$\Delta S = S - S_e,$$

is the change in entropy from its equilibrium value, which can be decomposed (Prigogine, 1967) in the form

$$\Delta S = \Delta_e S + \Delta_i S, \quad (\text{A.4})$$

where $\Delta_e S$ and $\Delta_i S$ are the changes (from equilibrium) in external entropy supply and internal entropy production, respectively.

The first law of thermodynamics can be written in the form (Fung, 1965)

$$du = \sigma_{ij} d\varepsilon_{ij} + T d_e S, \quad (\text{A.5})$$

where $d_e S$ is an infinitesimal change in external entropy supply. Integrating the above differential form from equilibrium to an arbitrary current state we get

$$u - u_e = \sigma_{ij} [\varepsilon_{ij} - (\varepsilon_{ij})_e] + T \Delta_e S. \quad (\text{A.6})$$

Substituting (A.6) into (A.4) and using (A.3) we obtain

$$\phi^v = \phi_e^v - T \Delta_i S. \quad (\text{A.7})$$

For fixed stress, damage, and temperature; the internal entropy production takes the form (Prigogine, 1967)

$$d_i S = \frac{\Gamma_r d\gamma_r}{T}. \quad (\text{A.8})$$

Following Prigogine (1967), expand Γ_r in terms of γ_r around the equilibrium values γ_r^e , noting that $\Gamma_r = 0$ at $\gamma_r = \gamma_r^e$ and keeping up to linear terms only, to get

$$\Gamma_r = \left(\frac{\partial \Gamma_r}{\partial \gamma_q} \right)_e (\gamma_q - \gamma_q^e) + \text{H.O.T.} \quad (\text{A.9})$$

Substituting (A.9) into (A.8) and integrating from equilibrium to the current state yields (Prigogine, 1967, pp. 47, 48)

$$T \Delta_i S = \frac{1}{2} \left(\frac{\partial \Gamma_r}{\partial \gamma_q} \right)_e (\gamma_r - \gamma_r^e) (\gamma_q - \gamma_q^e) + \text{H.O.T.} \quad (\text{A.10})$$

Clearly, substitution of (A.10) into (A.7) and using (22) gives the expansion in (31), which establishes the stated equivalence.

APPENDIX B

In this Appendix, we show that a special form of (46) can be viewed as a generalization of the modified superposition integral. To this end, consider the special case in which all γ_r^e , and hence all $\hat{\gamma}_r^e$ and Λ_r , have common dependence on stress, damage and temperature, i.e.

$$\Lambda_r(\sigma_{ij}, \omega_{abcd}, T) = c_r \lambda(\sigma_{ij}, \omega_{abcd}, T) \quad \forall r, \quad (\text{B.1})$$

where c_r is a scaling constant and λ is a scalar function of its arguments. Denoting the time-dependent part of the strain in (46) by ε_{ij}^c and using (B.1), we can write

$$\varepsilon_{ij}^c = \int_{0-}^t h(\xi - \xi') \frac{\partial}{\partial \tau} \left(\frac{\partial \lambda}{\partial \sigma_{ij}} \right) d\tau, \quad (\text{B.2})$$

where

$$h(\xi) = \sum_r c_r \left(1 - e^{-\xi/\tau_r} \right). \quad (\text{B.3})$$

Considering the isothermal undamaged material behavior and assuming that a is independent of σ_{ij} , then (B.2) becomes

$$\varepsilon_{ij}^c = \int_{0-}^t h(t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\partial \lambda}{\partial \sigma_{ij}} \right) d\tau, \quad (\text{B.4})$$

and λ becomes a function of the stress invariants only.

The meaning of the function $h(t)$ can be established by considering the response under sufficiently low stress levels. In this case the viscoelastic behavior is linear in stress, hence $\lambda \sim \sigma^2$, namely

$$\lambda = \frac{1}{2} \widehat{\Delta S}_{ijkl}^o \sigma_{ij} \sigma_{kl}, \quad (\text{B.5})$$

where $\widehat{\Delta S}_{ijkl}^o$ is a symmetric fourth rank tensor. Substitution of (B.5) into (B.4) gives

$$\varepsilon_{ij}^c = \int_{0-}^t \widehat{\Delta S}_{ijkl}(t - \tau) \frac{\partial \sigma_{kl}}{\partial \tau} d\tau, \quad (\text{B.6})$$

where

$$\widehat{\Delta S}_{ijkl}(t) = h(t) \widehat{\Delta S}_{ijkl}^o. \quad (\text{B.7})$$

Since, by hypothesis, the retardation times τ_r are independent of stress, then $\widehat{\Delta S}_{ijkl}(t)$ is the time-dependent compliance tensor. All components of this compliance tensor have common time dependence expressed by $h(t)$. It follows that $h(t)$, which is analogous to $\Delta D(t)$ in (94), is the linear uniaxial creep compliance of the material and can be determined from a uniaxial creep test under sufficiently low stress levels.

With the above interpretation for $h(t)$ and assuming that a is independent of σ_{ij} , i.e. $a = a(\omega_{abcd}, T)$, it is clear that (B.2) can be considered as a multiaxial generalization of

the modified superposition integral which also accounts for damage and thermal effects. This interpretation for (B.2) is, however, restricted to materials characterized by common time-dependence in all components of the creep compliance tensor in the linear range of behavior. For isotropic behavior, this corresponds to the case of materials exhibiting a constant Poisson's ratio, which applies for many polymers (Ferry, 1980).

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